Correlations, mean-field properties, and scaling of a one-dimensional sandpile model

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We present a general relationship between different scaling exponents for the one-dimensional sandpile problem to describe the self-adjusting of the slope of the sandpile. We solve the mean-field theory for this model, assuming that there is no correlation between the sizes of neighbor clusters. The mean-field theory does not give the correct exponents, since the clusters are strongly correlated. We characterize these correlations, identify the functional form of the cluster distribution function, and show how the multifractal scaling for averaged quantities arises from this form.

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A set of problems concerning nonlinear transport have attracted a great deal of attention lately. Soon after the introduction of the well-known two-dimensional (2D) sandpile model by Bak, Tang, and Wiesenfield [1] as an example of critical self-organization, a limited local (LL) 1D sandpile model was presented by Kadanoff *et al.* [2]. The original motivation to study this model was the hope of finding an easy solution, applicable to more complicated sandpile problems in higher dimensions. However, after long efforts by various groups [2–6], certain aspects of this model are still poorly understood. For example, the functional form of the distribution functions was not resolved. It was found that different distribution functions scale multifractally [2], and that the model has a set of nontrivial scaling exponents [3–5]. In [4], the dynamics of the model was claimed to be described by a singular diffusion equation. However, no direct connection was made between the "microscopic" properties of the system and this equation.

In this paper, we try to advance the understanding of this model in several ways. We first establish a set of relationships between the different local variables, describing the system. From this, one can construct a global phenomenology of how the system adjusts its slope and the cluster sizes close to their critical values. We next formulate a mean-field theory, assuming that the clusters are not correlated, and solve it completely. The mean-field results are not correct due to strong intercluster correlations. We go one step further, characterizing the cluster distribution function and the nontrivial correlations that arise in this model. We finally derive the multifractal scaling analytically.

The LL model is a simple 1D cellular automaton with a given integer value of "height of sandpile" h_i on each site of the size-L lattice. The system is perturbed by throwing sand on a randomly chosen site i:

$$h_i \rightarrow h_i + 1 \; . \tag{1}$$

If the slope $z_i = h_i - h_{i+1}$ becomes larger than 4, two grains of sand fall to the nearest downhill site:

$$\begin{aligned} h_i &\rightarrow h_i - 2 \; , \\ h_{i+1} &\rightarrow h_{i+1} + 2 \; . \end{aligned} \tag{2}$$

This process can generate avalanches, where other sites will become unstable, and topplings will occur in many sites. The left edge of the system is open to sand. The perturbation rate is very low, and perturbations are made only after the previous avalanche is finished.

It is sometimes convenient to use the slope variables. In term of slope, the perturbation rule is conservative:

$$\begin{aligned} z_i &\rightarrow z_i + 1 \; , \\ z_{i-1} &\rightarrow z_{i-1} - 1 \; . \end{aligned} \tag{3}$$

The toppling rules for the slope are the following. If $z_i > 4$, then

$$\begin{aligned} z_i &\rightarrow z_i - 4 \; , \\ z_{i\pm 1} &\rightarrow z_{i\pm 1} + 2 \; . \end{aligned} \tag{4}$$

A geometrical description of avalanche clusters is very useful. In this model, those clusters are defined by their boundary points, where $z_i \le 2$. Avalanches are always stopped at such sites. After [3], we call those sites *troughs.* The size of a cluster is the distance between two neighbor troughs.

We first present some general arguments about the mean properties of this model. Since each avalanche toppling changes the value of slope by an even number, a change in the oddness of the slope occurs only after perturbations [6]. Therefore, as the perturbations are random, the densities of sites with slopes of 3 and 4 are equal. To describe the local properties of the system, we use the average local slope $\langle z(x) \rangle$. The density of troughs at point x is proportional to $\epsilon(x) = 3.5 - \langle z(x) \rangle$. Notice that 3.5 is a critical slope, where the cluster and the avalanche size diverge. We expect all relevant variables of the system to diverge as powers of the critical distance $\epsilon(x)$. The average cluster size is proportional to $[\epsilon(x)]^{-1}$. The most important scaling is the scaling of the sand current, which is mediated through the avalanches. Since after each toppling two grains of sand are transported one step down, the current is proportional to the average number of topplings in an avalanche $\langle s(x) \rangle$. We assume that it scales as

$$\langle s(x) \rangle \sim \epsilon(x)^{-\gamma} \; , \tag{5}$$

where γ is a critical exponent.

Using this scaling, we can estimate some stable-state properties of the model and the way it self-adjusts the slope. The perturbation is done throughout the system, but sand can leave it only at the right edge. This implies a transport constraint on the local current. The average influx of sand to the left of a point x due to perturbations is x/L. It should be balanced by the average avalanche current through this point:

$$x \sim \langle s(x) \rangle \sim \epsilon(x)^{-\gamma} . \tag{6}$$

Hence, the average slope profile is

$$\epsilon(x) \sim x^{-1/\gamma} . \tag{7}$$

The cluster size will scale as $\langle \lambda(x) \rangle \sim x^{1/\gamma}$. Since the trough density is proportional to ϵ, the total number of troughs obeys

$$N \sim \int_0^L \epsilon(x) dz \sim L^{1-1/\gamma} . \tag{8}$$

The coarse-grained dynamics of this system was claimed to be governed by a singular diffusion equation [4]. To derive this limit, we first notice that the continuity equation for sand implies $\partial h(x,t)/\partial t = \partial \langle s(x,t) \rangle / \partial x$. Since the slope $z(x,t) = \partial h(x,t)/\partial x$, and the avalanche size scales with $\epsilon(x)$ as (5), we get

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[\frac{\text{const}}{[z_c - z(x,t)]^{-\gamma-1}} \frac{\partial z(x,t)}{\partial x} \right] . \tag{9}$$

This is exactly the phenomenological singular diffusion equation introduced in [4].

We now make a simple estimate for the value of γ. Two distinct types of avalanches occur in this model. If a pair of sites with slopes (4,4) is perturbed, the resulting avalanche will consist of multiple sliding of particles. For a distance r from the perturbation point to the nearest left trough, and a cluster size λ, the total number of topplings will be $r(\lambda - r)$. If the perturbed pair is (3,4), the avalanche will consist of a slide of two particles from the perturbed site to the nearest right trough. The number of topplings in such an avalanche is $\lambda - r$. Avalanches of the first type are dominant in the sand transport. Assuming that the probability to start such an avalanche is the same for every point inside the cluster, the average avalanche size will scale as λ^2. Since $\lambda \sim \epsilon^{-1}$, we estimate $\gamma = 2$. The numerical simulations contradict our estimate of γ and instead give $\gamma \approx 2.9$. We dedicate the rest of this paper to understanding the mechanisms for this anomalous scaling and the meaning of our $\gamma = 2$ approximation.

We found it useful to introduce a periodic version of the LL model. We call it the *Escher model* [8], since it resembles Escher's famous picture of a circular staircase. On such a fictitious staircase, moving in one direction will be a constant descent; conversely, in the opposite direction, it will be a constant ascent. The slope variable z_i is defined on a periodic lattice of size L. The toppling rules for slope are the same, but there is no longer any boundary.

The advantage of this model is its homogeneity. The total slope is conserved in all events, so there is no self-adjustment of the average slope. The global variable $\epsilon = 3.5 - \langle z \rangle$ is determined by the initial configuration and does not change with time. We can use this to study the scaling of different averages with ϵ. Numerical simulations give us

$$\begin{aligned} &\langle \lambda \rangle \sim \epsilon^{-1} , \quad \langle s \rangle \sim \epsilon^{-2.9} , \\ &\langle \lambda^2 \rangle \sim \epsilon^{-2.5} , \quad \langle \lambda^3 \rangle \sim \epsilon^{-3.7} . \end{aligned} \tag{10}$$

An example of such a scaling is shown in Fig. 1. Those results display multifractality, in a sense that $\langle \lambda^k \rangle \not\sim \epsilon^{-k}$. Multifractality was observed in a different context for the LL model with an open edge in previous papers [2,5]. The exponent γ, within the measurement mistakes, is the same as in the model with an open boundary [4–6].

The first step in investigating of the Escher model is to determine the average and fluctuations of the total number of troughs in the system. There are two types of basic events that change the number of troughs.

(1) *Creation,* when sand is thrown on the (3,3) pair. No avalanche will follow, and we will get (2,4), i.e., a new trough is born.

(2) *Coalescence,* when sand is thrown on the (4,4). There will be a large avalanche, destroying the nearest two troughs and creating one in between them. The total amount of troughs will be diminished by 1.

From the conservation of the total slope, we obtain the given number of troughs n:

$$\begin{aligned} \frac{N_3}{L} &= \tfrac{1}{2} + \epsilon - \frac{2n}{L} , \\ \frac{N_4}{L} &= \tfrac{1}{2} - \epsilon + \frac{2n}{L} , \end{aligned}$$

where N_3 and N_4 are the numbers of 3's and 4's all over the system. If ϵ and n/L are small and 3's and 4's are uncorrelated, the probabilities of having (3,3) and (4,4) are, respectively,

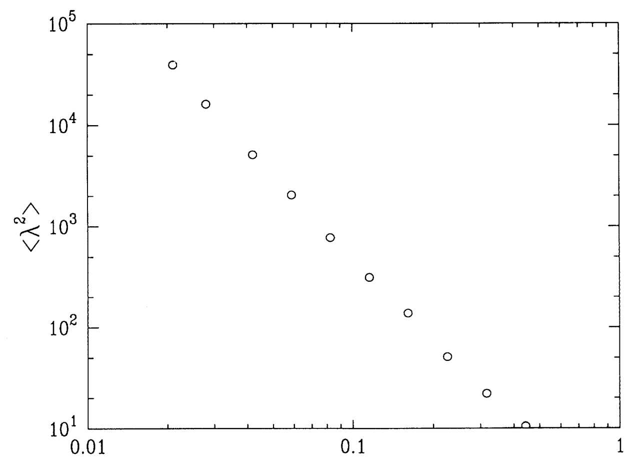

FIG. 1. As an example, we show the scaling of $\langle \lambda^2 \rangle$ vs ϵ. The scaling is very clean. The exponent is -2.5. The simulations were performed on a system of size $L = 4000$.

$$P(3,3)=\left[\frac{N_3}{L}\right]^2=\tfrac{1}{4}+\epsilon-\frac{2n}{L}\ ,$$

$$P(4,4)=\left[\frac{N_4}{L}\right]^2=\tfrac{1}{4}-\epsilon+\frac{2n}{L}\ .$$

So the master equation for the total number of troughs is

$$P_{t+1}\left[\frac{n}{L}\right]=\tfrac{1}{2}P_t\left[\frac{n}{L}\right]+\left[\frac{1}{4}+\epsilon-\frac{2n}{L}\right]P_t\left[\frac{n-1}{L}\right]+\left[\frac{1}{4}-\epsilon+\frac{2n}{L}\right]P_t\left[\frac{n+1}{L}\right]\ ,$$

where $P_t(x)$ is probability of having the density of troughs equal to x at time t. Expanding $P_t(n/L\pm 1/L)$ up to the second order in $1/L$, we get

$$P_{t+1}(x)-P_t(x)=\frac{1}{L}\left[(4x-2\epsilon)\frac{d}{dx}P_t(x)+\frac{1}{4L}\frac{d^2}{dx^2}P_t(x)\right]\ . \qquad (11)$$

The steady-state solution of this equation is

$$P(x)=\exp[-8L(x-\epsilon/2)^2]\ ,$$
$$p(n)=\exp\left[-\frac{8}{L}(n-\epsilon L/2)^2\right]\ , \qquad (12)$$

where $p(n)$ is the probability of having n troughs in the system.

The average density of troughs in the steady state is $\epsilon/2$. The average cluster size is

$$\langle\lambda\rangle=2/\epsilon\ . \qquad (13)$$

The trough number is sharply peaked at $\epsilon L/2$. Since fluctuations scale as $L^{1/2}$, they can be neglected in a large enough system. Numerical simulations confirm those predictions. It was already noted that the LL model displays a similar behavior [6]. However, as was mentioned by Krug, the width of this distribution scales as $L^{0.34}$. Notice that for both boundary conditions the trough number fluctuates as the square root of its average.

The next natural step is to introduce $P(\lambda)$, the probability of having a cluster of size λ. The scaling properties of the avalanches are determined by this distribution. The average avalanche size, for instance, can be found in the following way. The probability of hitting a cluster of size λ is proportional to λ. Therefore, the proper distribution function for averaging the avalanche size is $P_1(\lambda)=\lambda P(\lambda)/\langle\lambda\rangle$. Assuming that the probability of initiating an avalanche is the same for all points inside the cluster [7], i.e., that clusters have no internal structure, the scaling of the average avalanche is $\langle s\rangle=\langle\lambda^3\rangle/\langle\lambda\rangle$.

The master equation for this probability distribution can be derived using the basic mechanisms of trough creation and coalescence, and assuming equal probabilities for creation and coalescence events. Namely, the probability that a cluster of size λ will be split in two by the creation of a trough inside it is proportional to $\lambda P(\lambda)$. The probability that the cluster will be destroyed by the coalescence event is $[\lambda+2\bar{\lambda}(\lambda)]P(\lambda)$, where $\bar{\lambda}(\lambda)$ is the average size of the cluster neighboring λ. The probability that a cluster of size λ will be created by splitting a larger one is $2\sum_{\lambda'>\lambda}P(\lambda')$. And, finally the probability that a new cluster of size λ will be created as a result of a coalescence event, is $2\sum_{\lambda''<\lambda}^{\lambda'+\lambda''>\lambda}P(\lambda',\lambda'')$, where $P(\lambda',\lambda'')$ is the probability of having two neighboring clusters of sizes λ',λ''. Combining these probabilities into the master equation, we get

$$\frac{\partial P(\lambda)}{\partial t}=-\lambda P(\lambda)-[\lambda+2\bar{\lambda}(\lambda)]P(\lambda)+2\sum_{\lambda'>\lambda}P(\lambda')+2\sum_{\substack{\lambda'+\lambda''>\lambda\\ \lambda''<\lambda}}P(\lambda',\lambda'')\ . \qquad (14)$$

This master equation is quite complicated. To obtain some understanding of it, we assume that clusters are not correlated. From this mean-field assumption, we get

(1) $P(\lambda',\lambda'')=P(\lambda')P(\lambda'')$,

(2) $\bar{\lambda}(\lambda)$ does not depend on λ, and is equal to $\langle\lambda\rangle$.

Now the master equation is closed. It can easily be shown that the solution of it has the exponential form

$$P(\lambda)\sim\exp(-\lambda/\langle\lambda\rangle)\ . \qquad (15)$$

This implies that $\langle\lambda^k\rangle$ must scale like $\langle\lambda\rangle^k$, i.e., like ϵ^{-k}. Our previous estimate $\gamma=2$ is now understood to be the result of the *mean-field approximation.* It is interesting to observe that the same exponential distribution, known as the Poisson distribution, arises when we randomly throw n points to an interval of size L and look at the distribution of the intervals L cut by those points. Here, as in our model, $\langle\lambda\rangle=L/n$.

To check that our understanding of the mean-field model is correct, we simulated a version of the original Escher model, where, each time before, the addition of sand clusters had been randomly mixed. Possible spatial correlations between two nearby clusters were destroyed by hand. This model indeed has $\gamma=2$ and the Poisson distribution function $P(\lambda)\sim\exp(-\lambda/\langle\lambda\rangle)$.

The Escher model itself is much less trivial, as was indicated by the multifractal scaling of the averages. There we found the following functional form of $P(\lambda)$:

$$P(\lambda)\sim\exp[-b(\epsilon)\ln^S\lambda]\ , \qquad (16)$$

where $S\approx 3.9$ is very close to 4. To show this, in Fig. 2 we present the dependence of $\ln P(\lambda)$ versus $\ln\lambda$. This functional form fits very well for small enough ϵ. The distribution functions in the LL model can also be described by similar expressions with very close exponents. For example, the drop-number distribution is characterized by $S=4.2$.

All the scaling exponents can be derived from this functional form. To get an approximate analytical expressions for $\langle\lambda^k\rangle$, we use the saddle-point approach in calculation of the average $\int_0^\infty\lambda^k P(\lambda)d\lambda$. The expression under integral reaches its maximum at $\lambda_{\max,k}$, such that $k=Sb(\epsilon)\ln^{S-1}\lambda_{\max,k}$. Approximately, $\langle\lambda^k\rangle\approx\lambda_{\max,k}^k$.

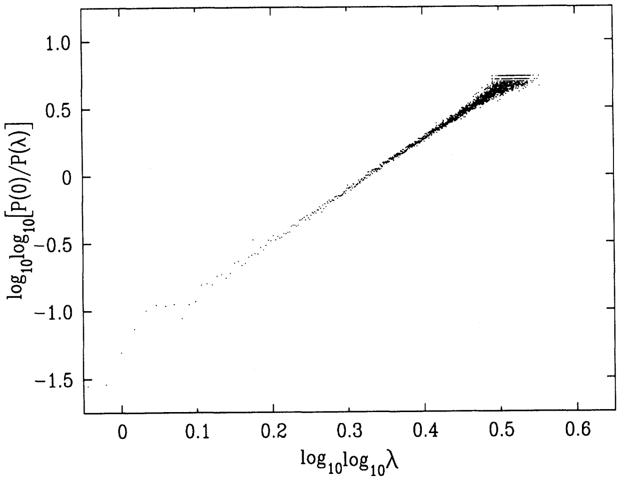

FIG. 2. The dependence $\log_{10}[P(0)/P(\lambda)] \sim \log_{10}^{S}\lambda$ is shown on the figure, using a double logarithmical scale. The exponent S is evaluated as 3.9. The system parameters are $L=8000$ and $\epsilon=0.025$.

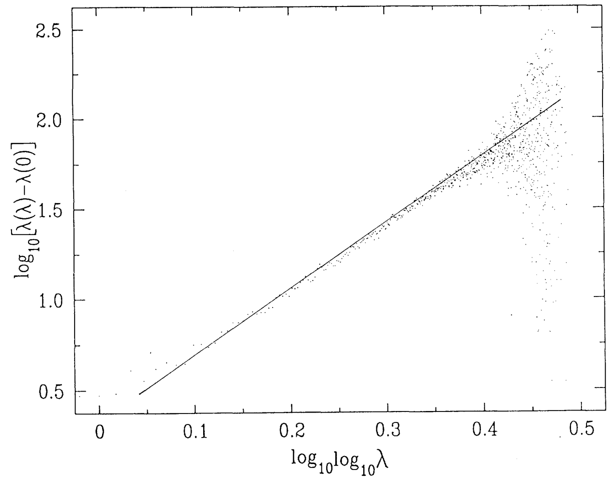

FIG. 3. The conditional average $\bar{\lambda}(\lambda)$ is shown to have the form $c+d\log_{10}^{S'}\lambda$, where $S'\approx 3.8$ is very close to the value of the exponent S from the previous figure. The system parameters are $L=4000$ and $\epsilon=0.05$.

Combining this with the constraint (13) that $\langle\lambda\rangle \sim \epsilon^{-1}$, we can estimate $b(\epsilon)=1/(\text{const}-\ln\epsilon)^{S-1}$. So we get

$$\langle\lambda^k\rangle \sim \epsilon^{-k^{S/S-1}} . \tag{17}$$

This is a reasonable estimate for the numerical results (10).

For example, for $k=2$, we get $\langle\lambda^2\rangle \sim \epsilon^{-2.5}$, which is in good agreement with the numerically calculated critical index.

The failure of the mean-field theory is obviously related to the correlations between neighboring clusters. To obtain further insight into the problem, we measure numerically the conditional average $\bar{\lambda}(\lambda)$.

Unlike the mean-field version, this is not a constant. The λ dependence has the following form:

$$\bar{\lambda}(\lambda)=c(\epsilon)+d(\epsilon)\ln^{3.8}\lambda . \tag{18}$$

The example of this dependence is given in Fig. 3. The clusters are obviously highly correlated: the larger the cluster, the larger will be its average environment.

To conclude, we found how the sandpile organizes the scale of its clusters to balance the current of sand. The local average cluster size changes with the position. Therefore, one cannot speak of a single length scale in this problem. We presented a closed mean-field theory, giving all the distribution functions and scaling exponents for the LL and Escher models. The contradiction between these exponents and those from the numerical simulations is related to the strong correlations in sizes of neighbor clusters. Finally, we have shown the correct form of the cluster distribution function. Using this simple analytic form, we can explain the multifractal properties of the model and the anomalous transport exponents observed for it.

We still do not understand why this form of correlation and distribution functions arises in this system. This is a subject of ongoing research. Another interesting question is whether the singular diffusion equation gives a proper description of the temporal behavior of this model. This is not clear, because the system needs time to build up the correlations, which are, as we know now, responsible for the anomalous γ. The initial studies of the subject verify our doubts.

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[7] Our numerical simulations confirm that this is true for large enough clusters. Generally, we have observed an exponential behavior for $P_4(x,\lambda)$—the probability to have 4 at a distance x from the left edge of cluster of size λ: $P_4(x,\lambda)=A\exp(-x/l)+B\exp[-(\lambda-x)/l]+C$. In this expression, A, B, C, and 1 are some constants, dependent on λ and ϵ. The larger the cluster, the flatter this profile.

[8] This name for the model was suggested by Per Bak.